Controlling spatiotemporal chaos in coupled nonlinear oscillators

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(Received 3 March 1997)

A method for controlling spatiotemporal chaos in coupled ordinary differential equations is presented. It is based on two ideas: stabilization of unstable periodic patterns embedded in spatiotemporal chaos, and perturbation of dynamical variables only at regular time intervals. [S1063-651X(97)13306-2]

PACS number(s): 05.45.+b, 64.60.Cn

Many nonlinear systems such as current-biased series arrays of Josephson junctions [1], dynamic arrays of nonlinear electrical circuits [2], discrete reaction-diffusion equations [3], and networks of neurons and cardiac pacemaker cells [4], to mention only a few, are modeled by an array of diffusively coupled oscillators or, in other words, coupled ordinary differential equations (CODE's). The most typical type of behavior encountered in large interconnections of chaotic oscillators is spatiotemporal chaos, where the observed dynamics exhibits chaotic properties both in time and space. However, in many applications, it is advantageous to avoid chaos. In this paper we propose a method for controlling spatiotemporal chaotic dynamics in CODE's.

Since the seminal paper by Ott, Grebogi, and Yorke (OGY) [5], there has been wide activity in the area of chaos control across many disciplines [6]. Control of spatiotemporal chaos in coupled map lattices (CML's) and systems described with partial differential equations (PDE's) has been considered recently [7–11]. In Refs. [7,8] it was shown that the complex spatiotemporal behavior in CML's can be eliminated in favor of a coherent state in which all elements are synchronized to a prescribed periodic orbit using the OGY method [5] and a linear feedback approach [12,13]. By injecting negative feedback at a certain $\boldsymbol{x}$-space point, Hu and He [9] successfully stabilized unstable steady states and controlled chaos in a one-dimensional nonlinear drift-wave equation driven by a sinusoidal wave. In [10], control of a class of spatialy extended systems was achieved via the stabilization of an active source of traveling waves. Lu, Yu, and Harrison [11] successfully demonstrated control of unstable roll patterns in a transversely extended three-level laser, using a time- and space-dependent feedback approach.

Recently [14], the taming of spatiotemporal chaos in CODE's was achieved by random variations of a single parameter along the array. However, as stressed by Strogatz [15], no one knows what complex periodic pattern will arise after disorder is introduced. The approach we propose in this Brief Report is based on two ideas: (i) stabilization of unstable periodic patterns embedded in spatiotemporal chaos, and (ii) perturbation of dynamical variables (not the system parameters) only at regular time intervals. To control chaos by stabilizing periodic orbits is a crucial idea of the OGY method, and it has been successfully applied in various real systems [16]. On the other hand, systems which are driven only at discrete time intervals in such a way that some components of the state vector are reset to new values were also investigated recently. For example, systems driven by noise were considered in Ref. [17] with important consequences in Monte Carlo applications. Recently, this type of driven system was applied successfully to synchronize chaos [18,19] and spatiotemporal chaos [20].

Generally speaking, control always means to influence the system in such a way that it performs in a desired way. Our control strategy is based on the application of controllers dispersed periodically in space with period P. Each of them perturbs the value of a single-state variable of the oscillator or ODE where it is connected with. For simplicity we assume that the perturbations are T periodic in time. The motivation for such a control is twofold: (i) to reduce the number of controlling points, and (ii) to make possible the control through *time-discontinuous* monitoring and influencing of the controlled state variables. Therefore, we consider a class of systems where only control of state variables is possible, and not control of system parameters. The type of control we used in this paper can be implemented experimentally as described in Ref. [21].

Let us consider a one-dimensional array of N diffusively coupled oscillators with periodic boundary conditions

$$\dot{\boldsymbol{u}}_i=\boldsymbol{f}(\boldsymbol{u}_i)+\boldsymbol{D}(\boldsymbol{u}_{i+1}-2\boldsymbol{u}_i+\boldsymbol{u}_{i-1})+\boldsymbol{G}\boldsymbol{v}(t), \qquad (1)$$

where $\boldsymbol{u}_i$, $i=1,\ldots,N$, are n-dimensional vectors, $\boldsymbol{D}=\mathrm{diag}(d_1,d_2,\ldots,d_n)$, and $\boldsymbol{G}=\mathrm{diag}(g_1,g_2,\ldots,g_n)$ are constant diagonal matrices, and $\boldsymbol{v}(t)$ is an n-dimensional input signal representing the influence of the controllers to the CODE's. In the following we assume that only one element of the matrix $\boldsymbol{G}$ is equal to 1, and all others are zero. In other words, only one variable, say the y variable, of the state vector of each oscillator can be monitored and/or controlled. Let $\boldsymbol{s}(t)$ be a solution for a single oscillator (clearly it is also a solution of the coupled oscillators without driving) that should be stabilized (the control goal). The input signal $\boldsymbol{v}(t)$ has the following form:

$$\boldsymbol{v}(t)=\boldsymbol{g}_T(\boldsymbol{s}(t),\boldsymbol{u}_i)\Delta_{i,P}, \qquad (2)$$

where $\boldsymbol{g}_T$ is a function describing the control law (see below), and T is a parameter playing the role of a period. The last term in Eq. (2) is a modification of the Kronecker symbol and denotes

$$\Delta_{i,P}=\begin{cases}1 & \text{for } i \bmod P=0\\ 0 & \text{for } i \bmod P\neq 0.\end{cases}$$

The controllers are located periodically in space with period P (since we consider periodic boundary conditions, we also assume that N/P is an integer). The function $\boldsymbol{g}_T$ can be described as follows [22]: the CODE (1) oscillates unforced and free from the controlling signal $\boldsymbol{s}(t)$, except for the equidistant times nT, when the y variables of the N/P oscillators are simultaneously forced to new values, that is,

$$y_i(nT)=s_y(nT), \quad i=P,2P,\ldots,N$$

where $s_y(t)$ is the y projection of the control goal $\boldsymbol{s}(t)$ at times nT. Thus we say that the CODE (1) is controlled by the sequence $\{s_y(nT)\}$. Our control strategy relies on the following.

Theorem: Consider the dynamical system

$$\dot{\boldsymbol{u}}=\boldsymbol{f}(\boldsymbol{u}), \tag{3}$$

and its solution $\boldsymbol{s}(t)$. Decompose the state vector of this system into two parts $\boldsymbol{w}$ and $\boldsymbol{v}$, and the vector field $\boldsymbol{f}$ into $\boldsymbol{f}_w$ and $\boldsymbol{f}_v$, respectively. Assume that the system

$$\dot{\boldsymbol{v}}=\boldsymbol{f}_v(\boldsymbol{s}_w(t),\boldsymbol{v})$$

is asymptotically stable when continuously driven by $\boldsymbol{s}_w(t)$. Then system (3) can be controlled with the sequence of impulses $\{\boldsymbol{s}_w(nT)\}$; that is, the solution of Eq. (3) approaches the goal $\boldsymbol{s}(t)$ as time goes to infinity, provided that the sampling period T is sufficiently small [23,24].

We stress here that the error $e(t)=\|\boldsymbol{u}(t)-\boldsymbol{s}(t)\|$ approaches zero *only if* $\boldsymbol{s}(t)$ is an *exact solution* of Eq. (3). If, for example, the control goal is an unstable periodic orbit which is only approximately known, then $e(t)$ is only close to zero and never reaches it.

Therefore, the better one knows the unstable periodic orbits, the closer the error $e(t)$ goes to zero. We emphasize that this effect is not caused by the sampling of the unstable periodic orbit, and it is a common phenomenon in all similar control methods. An example is the linear control feedback method $\dot{\boldsymbol{u}}=\boldsymbol{f}(\boldsymbol{u})+\boldsymbol{k}[\boldsymbol{s}(t)-\boldsymbol{u}]$, where $\boldsymbol{s}(t)$ is an unstable periodic orbit.

In CODE's in general, the following types (phases) of behavior are possible [25]: (i) *Coherent (synchronized) phase*: all oscillators synchronize. (ii) *Ordered phase*: groups of oscillators are clustered, and in each group a coherent phase appears. (iii) *Turbulent phase*: each oscillator operates in a chaotic regime, and all oscillators are completely desynchronized.

Now we show how the turbulent phase in CODE's can be eliminated in favor of coherent and/or ordered phases. For the numerical simulations presented in this paper, we use the Lorenz system, but similar results have been obtained with other dynamical systems. In the following, the values of the parameters in the Lorenz system are fixed to $\sigma=10$, $r=23$, and $b=1$. We also consider only y-drive throughout the paper, that is, $g_1=0$, $g_2=1$, and $g_3=0$.

In the first simulation, the control goal is a chaotic orbit of a single Lorenz oscillator. As a result of the control, the turbulent phase is replaced by the coherent chaotic phase. In the simulation we have used $d_1=d_3=0$, $d_2=2$ [26], $N=51$, $P=3$, and $T=0.01$. Figure 1(a) depicts the spatiotemporal evolution of array (1). This figure shows the gray-scaled values of the x coordinates as a function of the time t and the spatial coordinate i. The control is switched on at $t=32$, as denoted by a dash-dotted line in Fig. 1(b). Figure 1(b) shows the control error and the synchronization error

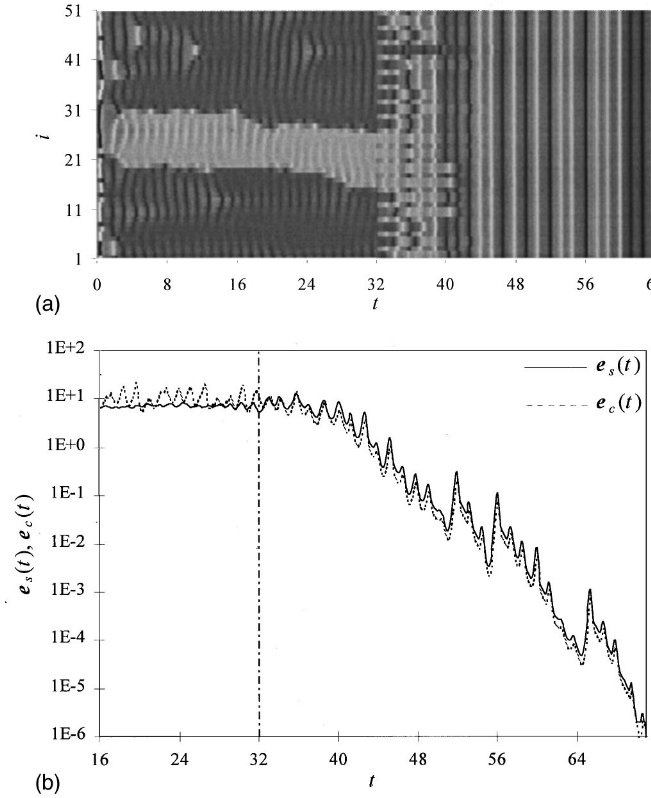

FIG. 1. Control of spatiotemporal chaos when the control goal is a coherent chaotic phase. $d_1=d_3=0$, $d_2=2$, $N=51$, $T=0.01$, and $P=3$. The control is switched on at $t=32$. (a) Gray-scaled x—coordinates of the array (1) as a function of the time t and the spatial coordinate i. (b) Control error $e_c(t)$ and synchronization error $e_s(t)$ vs time.

$$e_c(t)=\overline{\|\boldsymbol{u}_i(t)-\boldsymbol{s}(t)\|}, \quad i\in\{1,\ldots,N\}, \tag{4}$$

$$e_s(t)=\overline{\|\boldsymbol{u}_i(t)-\boldsymbol{u}_j(t)\|}, \quad i,j\in\{1,\ldots,N\}, \quad i\neq j \tag{5}$$

as functions of time, where $e_c(t)$ is averaged over all cells, while $e_s(t)$ is averaged over all pairs of distinct cells. Since both errors approach zero, one can conclude that the control of the coherent phase in the array is successfully achieved. We have also calculated numerically that for $d_2>1.15$ the array is asymptotically stable when control is applied. Therefore, the theorem provides additional evidence that the control in this case is successful.

In the second numerical experiment, $\boldsymbol{s}(t)$ is a periodic orbit embedded in the chaotic attractor of a single Lorenz oscillator. In other words, the control goal for CODE's is a coherent periodic phase. The unstable periodic orbit in the single Lorenz system is recovered using the technique proposed in Ref. [27], and its sampled values (with sampling

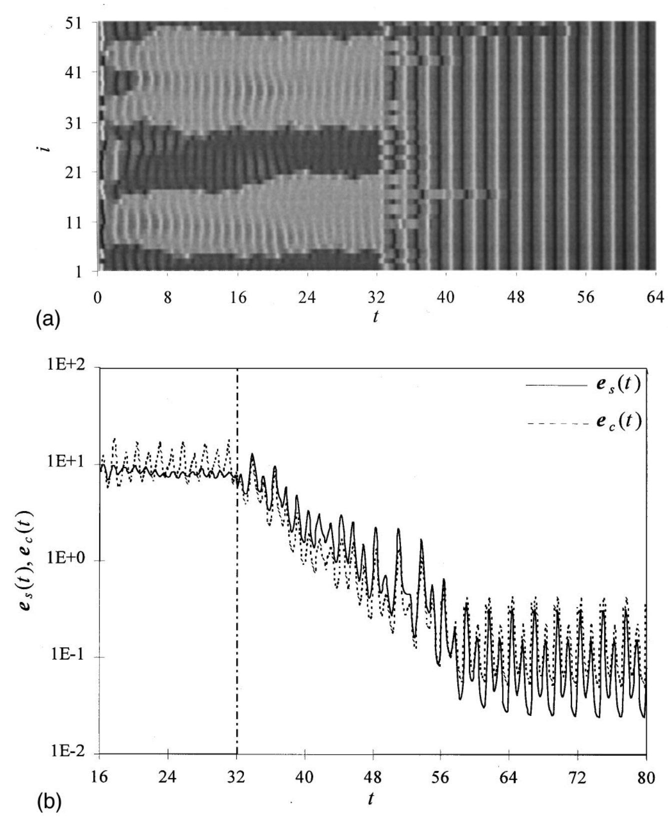

FIG. 2. Control of spatiotemporal chaos when the control goal is a coherent periodic phase. $d_1=d_3=0$, $d_2=2$, $N=51$, $T=0.01$, and $P=3$. The control is switched on at $t=32$. (a) Gray-scaled x—coordinates of the array (1) as a function of the time t and the spatial coordinate i. (b) Control error $e_c(t)$ and synchronization error $e_s(t)$ vs time.

period T) are stored in a controller [28]. The result is shown in Fig. 2. The parameter values are the same as in the previous simulation. Therefore, in this case the controlled array (1) is also asymptotically stable. However, after the control is switched on, the errors $e_c(t)$ and $e_s(t)$ defined by Eqs. (4) and (5) start decreasing, but they never approach zero [see Fig. 2(b)]. This is a numerical artefact already explained in the discussion following the theorem. The fact that the controlled array is asymptotically stable suggests that a functional relation between the control signal and the coordinates of the array exists. This is similar to the phenomenon of generalized synchronization [29].

We turn now to controlling ordered phases (or patterns) in CODE's. In this case the control goal is the following pattern: 00111100011110011100, where zeros and ones denote groups of five oscillators each, operating in a periodic regime with two different periodic orbits [30]. In this simulation we

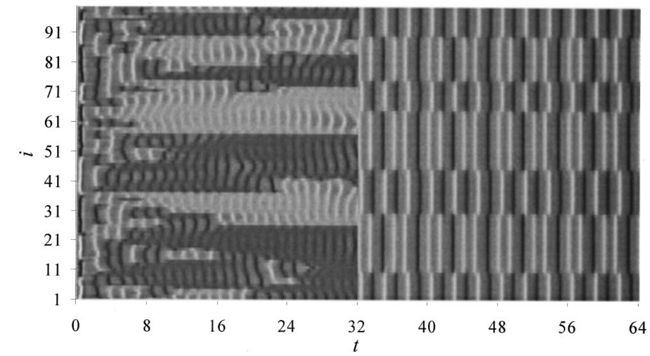

FIG. 3. Gray-scaled x—coordinates of the array (1) as a function of the time t and the spatial coordinate i. $d_2=d_3=0$, $d_1=6$, $N=100$, $T=0.01$, and $P=1$. The control goal is a pattern. The control is switched on at $t=32$.

use $d_1=6, d_2=d_3=0$, $N=100$, $P=1$, and $T=0.01$. The evolution of the uncontrolled array is chaotic with Lyapunov dimension equal to $D_L=69.3$. The other parameter values are as above. The results are shown in Fig. 3. Figure 3(a) shows the spatiotemporal evolution of the array. It is clearly visible that after the control is switched on, the ordered phase (pattern) is achieved almost immediately. The dependence of the control and the synchronization errors on the time is similar to that in Fig. 2(b), and is not shown here. In this case, these errors do not tend to zero (i) due to the coupling between the adjacent cells which are controlled by different periodic orbits, and (ii) due to the effect we described in the discussion after the theorem.

How general is the method for controlling spatiotemporal chaos in CODE's? We performed numerical experiments with different arrays of coupled cells. In all cases we observed similar results: it is possible to control coherent and/or ordered phases in CODE's via manipulating single variables of spatially separated elements and/or cells of the array at discrete times only. A more detailed analysis of the above results will be presented in an extended version of this work. We stress here that the method can also be used when the underlying dynamics is not known. This is based on the fact that the method can be viewed as a way to control the current state of the system with its prerecorded history. It does not require any analytical knowledge of the system dynamics, and can be simply implemented in experiments by a purely analog technique.

This work was supported by a binational German-Macedonian project (No. MAK-004-96) and by the Deutsche Forschungsgemeinschaft (Grant No. PA 643/1-1). P.J. acknowledges the support of the ICTP Associate Program, ICTP, Trieste, Italy.

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